The structure of a contact region, with application to the reflexion of a shock from a heat-conducting wall

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By using methods well known in boundary layer theory, the pressure across a contact region is shown to be approximately constant. A partial differential equation for the temperature is then derived. If the ideal-gas flow external to the contact region is known, the temperature profile can be determined. This can then be used to calculate successively the velocity and a better approximation for the pressure of the gas in the contact region. The theory is illustrated by obtaining the temperature, velocity and pressure distributions for a gas in a contact region moving with uniform velocity. The thermal conductivity of the gas is assumed to vary with the temperature τ like $k = k_n \tau^n$, where $n = 0$, 1 or 2. The results are valid for any temperature ratio across the region.

The general theory is also used to determine the motion of a plane shock which is reflected from a plane-conducting wall. The fluid between the reflected shock and the wall is at a higher temperature than that of the wall and a contact region adjacent to the wall results. Expressions for the temperature, velocity and pressure of the fluid are derived, and it is shown that the effect of heat conduction is to decrease the velocity of the reflected shock by an amount which varies as the inverse square root of the time.

1. Introduction

In ideal-gas theory, a contact region is represented by a discontinuity in the temperature and density but not in the pressure and gas velocity. In this paper the effect of viscosity and heat conduction on the internal structure of a contact region is discussed. By using methods well known in boundary layer theory the pressure across the contact region can be shown to be approximately constant. This enables an approximate equation for the temperature of the gas to be derived and, if the ideal-gas flow external to the contact region is known, the temperature profile to be determined. From this the velocity and a better approximation for the pressure of the gas can be successively calculated. These can then be used to determine a higher-order approximate equation for the temperature. Hall (1954) has considered the case of a contact region moving with uniform speed. He assumed the pressure to be constant across the contact region and checked this assumption by experiment, finding that very little pressure change occurs there. The present paper considers the general problem of an accelerating contact region, where the temperatures at its edges vary with time.

The theory is illustrated by determining the temperature, velocity and pressure distributions in a contact region moving with uniform velocity for gases, whose thermal conductivities vary with the temperature τ like $k = k_n \tau^n$, where $n = 0$, 1 or 2, and the k_n are constants. The results are valid for any temperature ratio across the contact region.

An examination of the distributions suggests that for more complicated problems, in which the replacement of the whole contact region by a discontinuity is no longer a good approximation, the part of the contact region adjacent to the low-temperature side could be regarded as a discontinuity behind which there was finite heat flux. Such discontinuities have been discussed by Fraser (1958) with reference to steady radiation fronts. He finds that two types exist, namely a radiation flame corresponding to a weak deflagration, and a radiation shock corresponding to a strong detonation. Hirschfelder, Curtiss & Bird (1954) have also discussed the structure of a radiation front moving into an opaque material in which the absorption is intense. The expression for the radiation flux uses the Rosseland mean free path for thermal radiation, and the expression is similar in form to that for the heat flux used in this paper with the 'conductivity' proportional to some power, n, of the temperature, where $n \geqslant 3$. Thus in the present paper, the solution for the case $n = 2$ serves as an indication to what happens in radiative heat transfer. For instance, it illustrates that the velocity of such a 'radiation front' relative to the fluid is small compared with the speed of sound and that the discontinuity is rarefactive. We can thus infer that for radiation in an opaque material a radiation flame will result. To find the structure of an unsteady radiation flame requires only a slight modification of the present treatment for a contact region.

The general theory is further illustrated by determining the flow set up when a plane shock hits a plane-conducting wall. The reflected shock is shown to be attenuated by the presence of the conducting wall.

2. General theory

A uniform gas of infinite extent is initially at rest. Heat is then supplied to the gas at a rate depending only on the distance y from some fixed plane $y = 0$ and the time t. If the gas initially in the region $y < 0$ is heated, then the equations governing the subsequent motion are

$$\frac{D\rho}{Dt} + \rho \frac{\partial u}{\partial y} = 0, \tag{1}$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial y} + \frac{4}{3}\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \tag{2}$$

$$c_p \frac{D\tau}{Dt} + \frac{\tau}{\rho^2}\left(\frac{\partial \rho}{\partial \tau}\right)_p \frac{Dp}{Dt} = Q(y,t)\, H(y_0 - y) + \frac{4}{3}\frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{\rho}\frac{\partial}{\partial y}\left(k \frac{\partial \tau}{\partial y}\right), \tag{3}$$

together with the equation of state

$$\rho = f(p, \tau), \tag{4}$$

where p, ρ, τ and u denote the pressure, density, temperature and gas velocity, respectively, μ and k are the coefficients of viscosity and thermal conductivity and are assumed to be functions of the temperature alone, $y_0(t)$ is the position at time t of the particle initially at the origin, $Q(y,t)$ is the rate at which energy is generated per unit mass of the fluid and $H(y_0-y)$ is the Heaviside unit function, which is zero for $y > y_0(t)$.

If the viscous and heat conduction terms are neglected, then the solution of the problem involves a shock wave which is propagated into the non-heated gas. The shock wave is followed by a contact surface separating the heated gas from the non-heated gas. The motion and conditions at the sides of both these discontinuities can be found given the rate of generation of energy in the fluid. This solution will be referred to as the 'ideal-gas' solution. In reality both the shock and the contact surface are regions of small but finite thickness, where the transition from one state to another takes place rapidly and where viscosity and heat conduction exert a dominating influence. Our attention will be confined to finding the structure of the contact region, assuming that the 'ideal-gas' solution is known.

Since a contact surface moves with the fluid, a Lagrangian frame of reference is convenient. Let x be the initial position of a particle, which is at position y at time t, and let $\bar{\rho}$ denote the initial constant density of the gas. Equation (1), which expresses the law of conservation of mass, can now be replaced by the integral expression

$$\bar{\rho}x = \int_{y_0(t)}^{y} \rho\, dy. \tag{5}$$

Now put $\psi = \bar{\rho}x$, which is by definition a constant for a particular particle, and choose ψ and t as the new independent variables. Equations (2) and (3) transform into

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial \psi} + \frac{4}{3}\frac{\partial}{\partial \psi}\left(\mu\rho\frac{\partial u}{\partial \psi}\right), \tag{6}$$

$$c_p\frac{\partial \tau}{\partial t} + \frac{\tau}{\rho^2}\left(\frac{\partial \rho}{\partial \tau}\right)_p \frac{\partial p}{\partial t} = Q(\psi,t)\,H(-\psi) + \tfrac{4}{3}\mu\rho\left(\frac{\partial u}{\partial \psi}\right)^2 + \frac{\partial}{\partial \psi}\left(k\rho\frac{\partial \tau}{\partial \psi}\right), \tag{7}$$

where $u = (\partial y/\partial t)_\psi$. The reader will note that the above transformation corresponds to the von Mises transformation of the boundary layer equations. If ρ is known as a function of ψ and t, equation (5) can be re-written in the form

$$y - y_0(t) = \int_0^{\psi} (1/\rho)\, d\psi, \tag{8}$$

which, on differentiating with respect to t keeping ψ constant, gives the velocity of a fluid particle

$$u = u_0(t) + \int_0^{\psi} \frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) d\psi, \tag{9}$$

where $u_0(t)$ is the velocity at time t of the particle initially at the origin.

Let P, ω and U denote the 'ideal-gas' solutions for the pressure, density and velocity. Let the actual pressure and gas velocity in the contact region be

$$p = P + p', \quad u = U + u'. \tag{10}$$

Substituting in equations (9) and (6), we obtain

$$u' = u_0'(t) + \int_0^{\psi} \frac{\partial}{\partial t}\left(\frac{1}{\rho} - \frac{1}{\omega}\right) d\psi, \tag{11}$$

$$\frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial \psi} + \frac{4}{3}\frac{\partial}{\partial \psi}\left[\mu\rho\left(\frac{\partial U}{\partial \psi} + \frac{\partial u'}{\partial \psi}\right)\right]. \tag{12}$$

We now proceed in a manner similar to that used to obtain the boundary layer equations and assume that the effective thickness δ of the contact region is small. Use suffices 1 and 2 to label quantities in the heated and non-heated parts of the gas adjacent to the contact region. Taking t as a quantity having magnitude of standard order, equation (11) shows that u' is $O\{(1-\omega_1/\omega_2)\,\delta\}$. Further, if in the contact region the diffusion terms in equation (7) are to be of the same magnitude as the remaining terms, then $(k\rho)/c_p = O(\delta^2)$, and if also $(\mu c_p)/k$ is at most of order unity, it follows from equation (12) that $\partial p'/\partial\psi$ is $O\{(1-\omega_1/\omega_2)\,\delta\}$. Hence, the total change in p' across the contact region is $O\{(1-\omega_1/\omega_2)\,\delta^2\}$, so that, if terms of this order are negligible, the pressure in equation (7) can be replaced by its 'ideal-gas' value, $P(\psi, t)$, plus an arbitrary function of the time which is $O\{(1-\omega_1/\omega_2)\,\delta\}$. The viscous term in equation (7) can be neglected also. The energy equation then becomes an equation for the temperature alone to be solved subject to the boundary condition at the edges of the contact region. In order to simplify the analysis, terms which are $O\{(1-\omega_1/\omega_2)\,\delta\}$ will be neglected; the pressure in equation (7) can then be replaced by the 'ideal-gas' pressure evaluated at $\psi = 0$, namely $P_0(t)$. Similarly, if the rate of generation of heat in the fluid is not too strongly dependent upon the temperature, $Q(\psi, t)$ can also be replaced by $Q_0(t)$. If this is not the case, then the temperature dependence must be retained and the analysis is more complicated. We also assume that the gas is perfect with constant specific heats. Equation (7) then becomes

$$\frac{\partial \tau}{\partial t} - \frac{\gamma-1}{\gamma P_0}\frac{dP_0}{dt}\tau = \frac{Q_0(t)\,H(-\psi)}{c_p} + \frac{P_0}{c_p R}\frac{\partial}{\partial \psi}\left(\frac{k}{\tau}\frac{\partial \tau}{\partial \psi}\right), \tag{13}$$

which is an equation for the temperature and is solved subject to the boundary conditions $\tau = T_1$ at $\psi = -\infty$, $\tau = T_2$ at $\psi = \infty$, where T_1 and T_2 are the temperatures of the gas on either side of the contact discontinuity. They satisfy the equations

$$\frac{dT_1}{dt} - \frac{(\gamma-1)}{\gamma P_0}\frac{dP_0}{dt}T_1 = \frac{Q_0(t)}{c_p}, \quad \frac{dT_2}{dt} - \frac{(\gamma-1)}{\gamma P_0}\frac{dP_0}{dt}T_2 = 0. \tag{14}$$

Substituting $\tau = T_1\theta(\psi, t)$ in equation (13), and using equations (14), we obtain

$$\frac{\partial}{\partial t}\left[\frac{T_1}{T_2}(\theta - H(-\psi))\right] = \frac{P_0 T_1}{c_p R T_2}\frac{\partial}{\partial \psi}\left(\frac{k}{\tau}\frac{\partial \theta}{\partial \psi}\right). \tag{15}$$

If $k = k_1\tau$, where k_1 is a constant, a slight change in the time variable reduces the equation to the well-known heat conduction equation. When k is some other function of the temperature numerical methods have to be employed. The two cases when k is a constant and when it is proportional to the square of the temperature are dealt with in § 3.

Once the temperature has been found as a function of ψ and t, the position y of a particle can be determined from equation (8), which can be written

$$y-y_0(t) = (R/P_0)\int_0^{\psi} \tau\, d\psi. \tag{16}$$

The temperature distribution in the contact region can then be plotted.

Further, by using equations (13), (11) and (12), the velocity and the pressure can be expressed as

$$u-U = -\frac{1}{\gamma P_0}\frac{dP_0}{dt}(y-Y)+\frac{k}{c_p\tau}\frac{\partial\tau}{\partial\psi}+f(t), \tag{17}$$

$$p-P = (\tfrac{4}{3}\sigma-1)\frac{k}{c_p\tau}\frac{\partial\tau}{\partial t}+\frac{1}{c_pP_0}\frac{dP_0}{dt}\int^{\tau}\left(\frac{k}{\gamma\tau}-\tfrac{4}{3}\sigma\frac{dk}{d\tau}\right)d\tau$$
$$+\left[\frac{d}{dt}\left(\frac{1}{\gamma P_0}\frac{dP_0}{dt}\right)-\left(\frac{1}{\gamma P_0}\frac{dP_0}{dt}\right)^2\right]\int^{\psi}(y-Y)\,d\psi-f'(t)\,\psi+\frac{1}{\gamma P_0}\frac{dP_0}{dt}f(t)\,\psi+g(t), \tag{18}$$

where $\sigma = (\mu c_p)/k$, $f(t)$ and $g(t)$ are arbitrary functions of the time, and Y is the position of a particle at time t given by 'ideal-gas' flow theory, this particle being at position y when the effects of viscosity and heat conduction are considered. Y is related to y by the approximate equation

$$\psi = \int_{y_0}^{y}\rho\, dy = \omega_1(Y-Y_0)\,H(Y_0-Y)+\omega_2(Y-Y_0)\,H(Y-Y_0), \tag{19}$$

where ω_1 and ω_2 are the values of the density on either side of the contact discontinuity where the gas is assumed to be ideal.

The functions $f(t)$ and $g(t)$ are determined by considering the effect the contact region has on the external ideal-gas flow. The obvious question to ask is to what order of approximation can the flow inside the contact region be fitted to the gas flow outside without change from that described by the 'ideal-gas' solution. For this we require that $y \to Y$, $u \to U$ and $p \to P$ at *both* edges of the contact region. Now just outside the contact region $\partial\tau/\partial\psi$ and $\partial\tau/\partial t$ are at most $O(1)$. Equation (17), in which terms of $O(\delta^2)$ are neglected, shows that the first two requirements are satisfied if $f(t)$ is identically zero. Equation (18) shows that to $O(\delta)$, $p = P$ everywhere in the contact region if $g(t)$ is $O(\delta^2)$, so that the third requirement can be satisfied. Hence the external gas flow is unaltered to $O(\delta)$. Equation (18) with $f(t) \equiv 0$ can now be used to determine the path of any particular particle in the contact region. However, to determine the pressure from equation (18), it is necessary to know $g(t)$ to $O(\delta^2)$ and this can only be found by considering the effect of the contact region on the gas flow external to it. In one case only can $g(t)$ be determined easily and this is for the case of a uniformly moving contact surface for which $dP_0/dt = 0$ and $\partial\tau/\partial t \to 0$ at the edges of the contact region. Equation (18) then shows that p can tend to P_0 at both edges of the contact region if $g(t) \equiv 0$. The effect of the contact region on the external flow is then $O(\delta^3)$. Once the pressure and velocity of the gas have been determined from equations (17) and (18), a higher order approximation for the temperature could be worked out.

3. The structure of a contact region moving with constant velocity

In this case the ratio of the temperatures at the edges of the region remains constant. The thermal conductivity of the gas is assumed to vary with the temperature like $k = k_n \tau^n$, $n = 0$, 1 or 2. Hall (1954) has already treated the case $n = 1$ and obtained results for the case $n = 0$ by assuming small temperature changes across the contact region. The same assumption will not be made here, so that the results are applicable for any temperature ratio.

(i) $k = k_0$.

This case is best treated by transforming equation (15) back into the Eulerian system and making use of equation (17). Equation (15) becomes

$$\frac{\partial \theta}{\partial t}+\left[u_0-\frac{k_0}{c_p\omega_1}\left(\frac{\partial \theta}{\partial y}\right)_0\right]\frac{\partial \theta}{\partial y}=\frac{k_0}{c_p\omega_1}\left[\theta\frac{\partial^2\theta}{\partial y^2}-\left(\frac{\partial \theta}{\partial y}\right)^2\right]. \tag{20}$$

A similarity solution of the above equation depending upon the variable $\eta = \alpha_0[y-y_0(t)]/\sqrt{t}$ is now sought. Choosing $\alpha_0 = \sqrt{\{(\omega_1 c_p)/k_0\}}$, equation (20) becomes

$$-\left[\tfrac{1}{2}\eta+\left(\frac{d\theta}{d\eta}\right)_0\right]\frac{d\theta}{d\eta}=\theta\frac{d^2\theta}{d\eta^2}-\left(\frac{d\theta}{d\eta}\right)^2. \tag{21}$$

Now divide equation (21) by $d\theta/d\eta$ and then differentiate with respect to η. Substituting $\theta = \phi^2$ in the resulting equation and multiplying the equation by ϕ''/ϕ' we obtain on integration,

$$\frac{\phi\phi''}{\phi'}=\pm\sqrt{\{\phi'^2-\log_e\phi'+\kappa\}}, \tag{22}$$

where κ is a constant and the dash indicates differentiation with respect to η. Equation (22) integrates again to give

$$\log_e\frac{\phi}{\phi_c}=\pm\int_{\phi_c'}^{\phi'}\frac{d\phi'}{\sqrt{\{\phi'^2-\log_e\phi'+\kappa\}}}, \tag{23}$$

where $\phi' = \phi_c'$ when $\phi = \phi_c$. Substitution of equation (22) into (21) yields the expression for the similarity variable

$$\eta = \mp 2\phi\sqrt{\{\phi'^2-\log_e\phi'+\kappa\}}+2\phi\phi'-4\phi_0\phi_0'. \tag{24}$$

The signs are fixed by considering conditions at infinity where ϕ' becomes zero. This shows that the upper signs in expressions (23) and (24) are taken for $\eta < \eta_c$ and the lower signs for $\eta > \eta_c$, where η_c is the value of η for which

$$\sqrt{\{\phi_c'^2-\log_e\phi_c'+\kappa\}}=0, \tag{25}$$

thus determining κ in terms of ϕ_c'.

Now as $\eta \to \infty$, $\phi \to \sqrt{(T_2/T_1)}$, $\phi' \to 0$; hence

$$\log_e\frac{1}{\phi_c}\sqrt{\frac{T_2}{T_1}}=-\int_{\phi_c'}^{0}\frac{d\phi'}{\sqrt{\{\phi'^2-\phi_c'^2-\log_e(\phi'/\phi_c')\}}}. \tag{26}$$

Similarly, as $\eta \to -\infty$, $\phi \to 1$, $\phi' \to 0$, so that

$$\log_e\frac{1}{\phi_c}=\int_{\phi_c'}^{0}\frac{d\phi'}{\sqrt{\{\phi'^2-\phi_c'^2-\log_e(\phi'/\phi_c')\}}}. \tag{27}$$

Hence it is seen that $\phi_c^2 = \sqrt{(T_2/T_1)}$ or $\tau_c = \sqrt{(T_1 T_2)}$, i.e. at $\eta = \eta_c(+0)$ the temperature is the geometric mean of the temperatures at $\pm\infty$.

Equation (23) can be integrated numerically given ϕ_c', which corresponds to fixing the temperature ratio T_1/T_2; for instance, $\phi_c' = 1/\sqrt{2}$ corresponds to $T_1/T_2 = \infty$. The temperature is then found as a function of ϕ'. Equation (24) can now be used to find η as a function of ϕ', thus enabling the temperature distribution to be plotted (see figures 1 and 2).

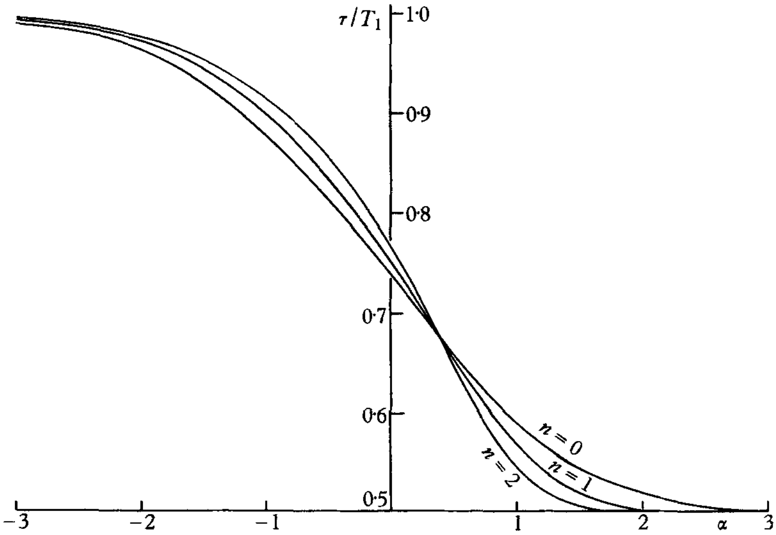

FIGURE 1. The temperature ratio τ/T_1 as a function of the similarity variable $\alpha = \{y - y_0(t)\}\,[(c_p\omega_1)/(k_n T_1^n t)]^{\frac{1}{2}}$ for $n = 0$, 1 and 2, and $T_2/T_1 = \frac{1}{2}$.

(ii) $k = k_1\tau$.

The solution of equation (15) for this case is well known and is

$$\tau = \tfrac{1}{2}(T_1 + T_2) - \tfrac{1}{2}(T_1 - T_2)\,\mathrm{erf}\,\tfrac{1}{2}\eta, \tag{28}$$

where $\eta = \sqrt{[(c_p R)/(P_0 k_1 t)]}\,\psi$. By substituting equation (28) into equation (16), $\sqrt{[(c_p\omega_1)/(k_1 T_1 t)]}\,(y - y_0(t))$ can be determined as a function of η. The temperature distribution can then be plotted (see figures 1 and 2).

(iii) $k = k_2\tau^2$.

Equation (15) becomes

$$\frac{\partial\theta}{\partial t} = \frac{k_2 P_0 T_1}{c_p R}\frac{\partial}{\partial\psi}\left(\theta\frac{\partial\theta}{\partial\psi}\right). \tag{29}$$

The reader will note that this equation is of the same form as that used to determine the velocity distribution in a laminar boundary layer of incompressible fluid. The problem now under consideration is analogous to the laminar mixing of two parallel streams, the velocity in this case being identified with the temperature in our problem.

We look for a similarity solution depending on the variable $\eta = \alpha_2\psi/\sqrt{t}$. Choosing $\alpha_2 = \sqrt{[(c_p R)/(P_0 k_2 T_1)]}$, equation (29) becomes

$$-\tfrac{1}{2}\eta\frac{d\theta}{d\eta} = \frac{d}{d\eta}\left(\theta\frac{d\theta}{d\eta}\right). \tag{30}$$

Putting $\theta = d\eta/d\zeta$ and taking ζ as the new independent variable, we obtain the Blasius equation

$$2\frac{d^3\eta}{d\zeta^3}+\eta\frac{d^2\eta}{d\zeta^2} = 0. \tag{31}$$

This has been solved numerically by Lock (1951) for the case of laminar mixing of two parallel streams for various ratios of the velocities. These results can be applied directly to obtain the temperature distribution for various values of the ratio T_1/T_2.

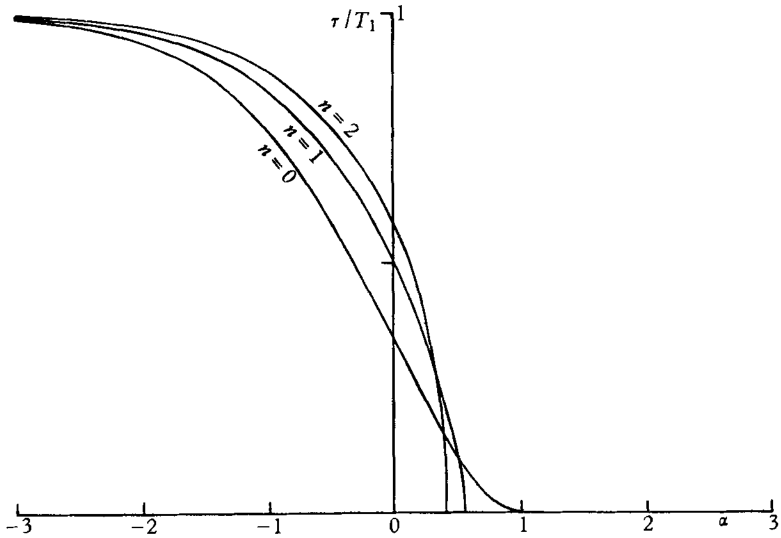

FIGURE 2. The temperature distribution for $T_2/T_1 = 0$ (with the same abscissa as in figure 1).

Results

Solutions have been obtained for the cases $T_2/T_1 = \frac{1}{2}$ and 0. The temperature distributions are plotted in figures 1 and 2 for $n = 0$, 1 and 2. They suggest that for more complicated problems, where the contact region could not be represented by a discontinuity, a theory could be developed in which that part of the contact region adjacent to the low temperature side could be replaced by a discontinuity behind which there was finite heat flux. The velocity of such a discontinuity relative to the fluid would in general be small compared to the speed of sound. The velocity and pressure distributions are shown in figures 3 to 6. They show that if the above approximate replacement was made, then the discontinuity would be rarefactive and would correspond to a weak deflagration.

Figure 2 shows that when $T_2/T_1 = 0$ the temperature gradient is infinite at the low temperature end for $n = 1$ and 2. This is due to assuming zero temperature there, thus introducing a singularity into equations (13), (17) and (18). In an actual case, where the temperature is small but finite, the profile would be smooth. The same singular behaviour is exhibited in the velocity and pressure

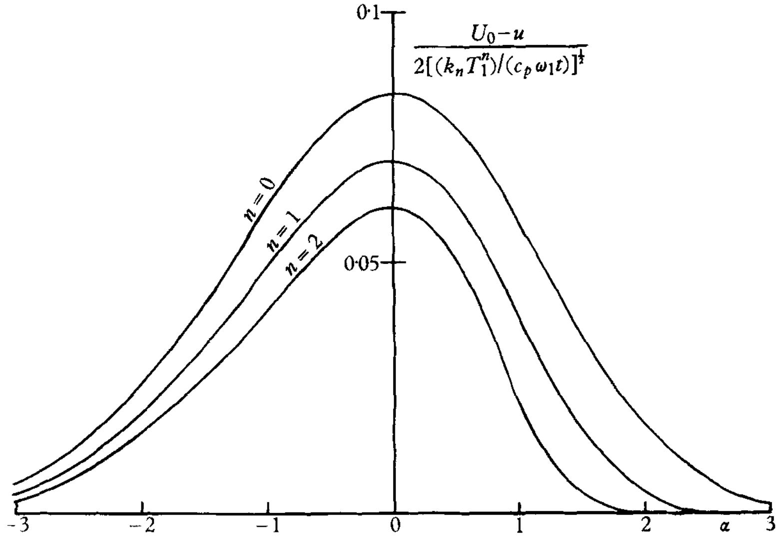

FIGURE 3. The perturbed velocity distribution for $T_2/T_1 = \frac{1}{2}$ (U_0 is the 'ideal-gas' velocity, defined in a similar way to P_0).

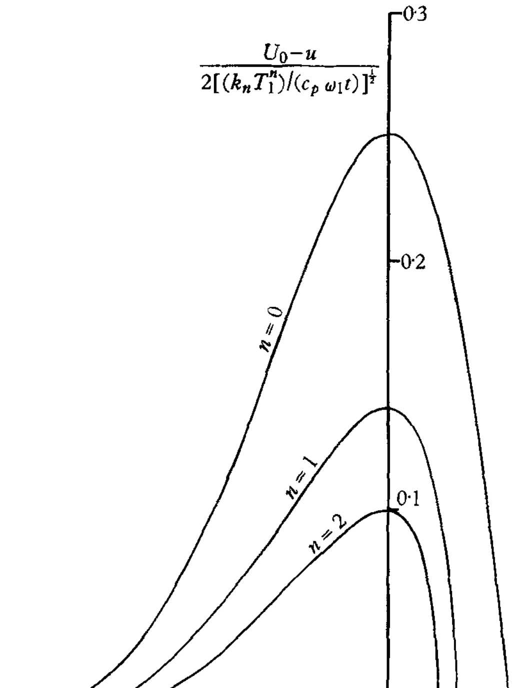

FIGURE 4. The perturbed velocity distribution for $T_2/T_1 = 0$.

diagrams. The pressure distribution for $T_2/T_1 = 0$ and $n = 0$ shows a discontinuity at the low temperature end. This is due to the non-vanishing of the term $(1/\tau)(\partial\tau/\partial t)$ in equation (18) as $\tau \to \tau_c$ ($= 0$ in this case). This discontinuity would not be present if T_2/T_1 was small but finite.

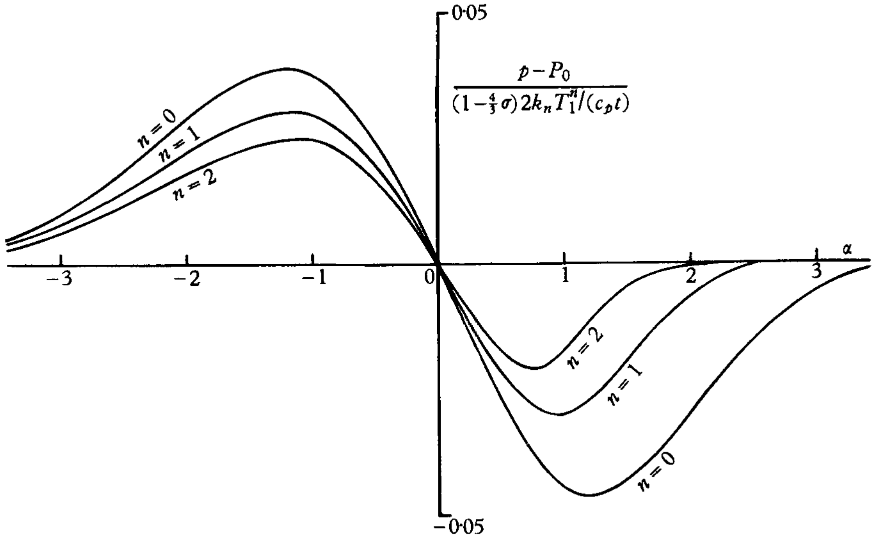

FIGURE 5. The perturbed pressure distribution for $T_2/T_1 = \frac{1}{2}$.

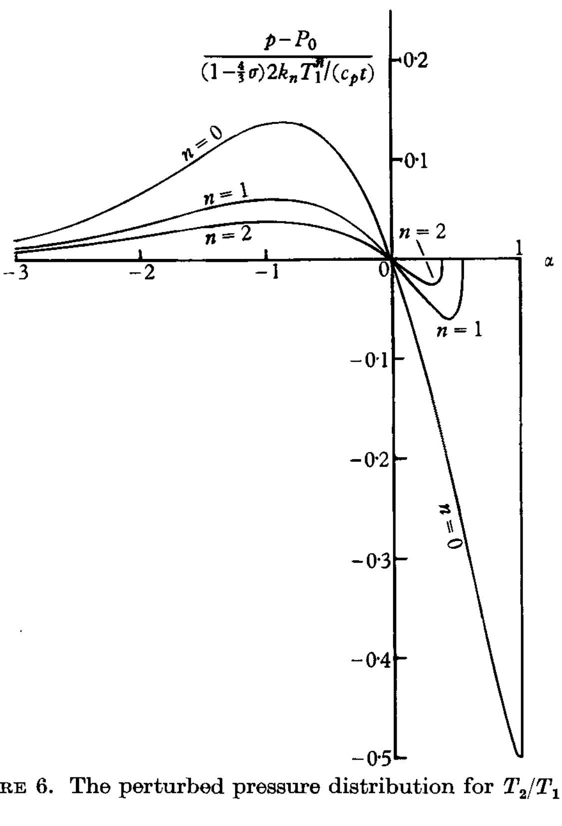

FIGURE 6. The perturbed pressure distribution for $T_2/T_1 = 0$.

4. The normal reflexion of a shock from a plane-conducting wall

At time $t = 0$, a plane shock of given strength is reflected from the face, $y = 0$, of a wall occupying the region $y > 0$. If viscosity and heat conduction are neglected, the velocity of the reflected shock can be determined. Let this velocity be U_s and let subscripts 2 and 3 indicate flow data ahead and behind the reflected shock respectively. The initial temperature of the wall is denoted by T_1. Near the face of the wall there will be a contact region in which heat conduction has an important effect and where we shall determine the temperature distribution.

In the region $y > 0$, the temperature satisfies the equation

$$\frac{\partial \tau}{\partial t} = \frac{k_w}{\rho_w c_w} \frac{\partial^2 \tau}{\partial y^2}, \tag{32}$$

where k_w is the thermal conductivity, ρ_w is the density, and c_w is the specific heat of the wall.

In the gas (in the region $y < 0$) adjacent to the wall, the temperature satisfies the equation [see equation (15)]

$$\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial \psi}\left(\frac{kP_3}{c_p R\tau} \frac{\partial \tau}{\partial \psi}\right). \tag{33}$$

Equations (32) and (33) are now solved subject to the boundary conditions $\tau \to T_1$ at the edge of the contact region in the wall, and $\tau \to T_3$ at the edge of the contact region in the gas. At the wall, the temperature and the flux of heat are continuous; thus

$$\left(k\frac{\partial \tau}{\partial y}\right)_{y=-0} = \left(k_w \frac{\partial \tau}{\partial y}\right)_{y=+0}. \tag{34}$$

To simplify the analysis we assume that $k = k_1 \tau$. Equations (32) and (33) can then be solved and yield the solutions

$$\tau - T_1 = A\left[1 - \operatorname{erf}\frac{y}{2\sqrt{\kappa_w t}}\right] \quad (\text{in } y > 0), \tag{35}$$

$$\tau - T_3 = B\left[1 + \operatorname{erf}\frac{\psi}{2\sqrt{\kappa_g t}}\right] \quad (\text{in } y < 0), \tag{36}$$

where $\kappa_w = k_w/(\rho_w c_w)$ and $\kappa_g = (k_1 P_3)/(c_p R)$. Applying the conditions at $y = 0$, we obtain

$$A = \frac{T_3 - T_1}{1 + (1/m)}, \quad B = -\frac{T_3 - T_1}{1 + m}, \tag{37}$$

where $m^2 = (k_1 T_3 c_p \omega_3)/(k_w \rho_w c_w)$. In the gas the Eulerian co-ordinate y is determined by substituting expression (36) in equation (16). This gives

$$(1+m)\frac{\omega_3 y}{2\sqrt{\kappa_g t}} = \left(m + \frac{T_1}{T_3}\right)\frac{\psi}{2\sqrt{\kappa_g t}} - \left(1 - \frac{T_1}{T_3}\right)\left\{\frac{\psi}{2\sqrt{\kappa_g t}} \operatorname{erf}\frac{\psi}{2\sqrt{\kappa_g t}} + \frac{1}{\sqrt{\pi}}\left(e^{-(\psi^2/4\kappa_g t)} - 1\right)\right\}. \tag{38}$$

To find the velocity distribution in the gas, use is made of expression (17). The function $f(t)$ is determined from the condition that the particle velocity must be zero at the wall; hence

$$f(t) = -\left(\frac{k_1}{c_p}\frac{\partial\tau}{\partial\psi}\right)_0, \tag{39}$$

and therefore

$$u = \frac{k_1}{c_p}\frac{\partial\tau}{\partial\psi} - \left(\frac{k_1}{c_p}\frac{\partial\tau}{\partial\psi}\right)_0, \tag{40}$$

where τ is given by expression (36). Outside the contact region $\partial\tau/\partial\psi \to 0$, and the particle velocity there is

$$u_\infty = -\left(\frac{k_1}{c_p}\frac{\partial\tau}{\partial\psi}\right)_0 = -\sqrt{\frac{k_1 T_3}{c_p \omega_3 \pi t}}\left(\frac{1-T_1/T_3}{1+m}\right). \tag{41}$$

To determine the pressure of the gas in the contact region, we need to know the function $g(t)$ in expression (18). This function has now to be calculated by perturbing the flow external to the contact region.

5. Solution in the external ideal-gas flow

We assume that the effect on the external flow is small. This is true for $t \gg k_1/(c_p \omega_3 R)$, which is of the order of the time when the shock is at a distance from the wall comparable to the molecular mean free path. We now write

$$u = u', \quad \rho = \omega_3 + \rho', \quad p = P_3 + p', \tag{42}$$

and substitute in the ideal-gas flow equations. By neglecting squares and higher powers of the perturbed quantities we can obtain the wave-equation for u'; thus

$$u' = F(A_3 t + y) + G(A_3 t - y). \tag{43}$$

Also

$$p' = -\omega_3 A_3[F(A_3 t + y) - G(A_3 t - y)]. \tag{44}$$

Linearizing the boundary condition at the edge of the contact region, i.e. applying the condition $u = u_\infty$ at the wall $y = 0$, we obtain

$$F(A_3 t) + G(A_3 t) = -\frac{K}{\sqrt{t}}, \tag{45}$$

where

$$K = \sqrt{\frac{k_1 T_3}{c_p \omega_3 \pi}}\left(\frac{1-T_1/T_3}{1+m}\right).$$

In a similar manner we apply the boundary condition at the unperturbed shock position given by $y = -U_s t$. It can be shown that here the velocity and pressure perturbations are related by the equation

$$p' = -\omega_3 A_3 \Phi(M_s)\, u', \tag{46}$$

where $M_s = U_s/A_3$ and

$$\Phi(M_s) = \frac{2M_s[(\gamma-1)M_s^2+2]}{[(3\gamma-1)M_s^2+3-\gamma]}. \tag{47}$$

Substituting expressions (43) and (44) in equation (46), we obtain

$$F(\overline{A_3 - U_s t}) = -\left(\frac{1+\Phi(M_s)}{1-\Phi(M_s)}\right) G(\overline{A_3 + U_s t}), \tag{48}$$

or

$$F(\xi) = -RG(\lambda\xi), \tag{49}$$

where ξ is a variable, $R = \{1+\Phi(M_s)\}/\{1-\Phi(M_s)\}$ and $\lambda = (1+M_s)/(1-M_s)$. Now put $\xi = A_3 t$ in equation (45) and substitute for $F(\xi)$ from expression (49), the following equation for G is then obtained,

$$-RG(\lambda\xi)+G(\xi) = -K\sqrt{(A_3/\xi)}, \tag{50}$$

which has the solution

$$G(\xi) = \frac{K}{(R/\sqrt{\lambda})-1}\sqrt{\frac{A_3}{\xi}}. \tag{51}$$

Using equation (49) we obtain

$$F(\xi) = -\frac{K(R/\sqrt{\lambda})}{(R/\sqrt{\lambda})-1}\sqrt{\frac{A_3}{\xi}}. \tag{52}$$

Hence
$$u' = -\frac{K}{(R/\sqrt{\lambda})-1}\left[\frac{R/\sqrt{\lambda}}{\sqrt{[t+(y/A_3)]}}-\frac{1}{\sqrt{[t-(y/A_3)]}}\right] \tag{53}$$

and
$$p' = \frac{\omega_3 A_3 K}{(R/\sqrt{\lambda})-1}\left[\frac{R/\sqrt{\lambda}}{\sqrt{[t+(y/A_3)]}}+\frac{1}{\sqrt{[t-(y/A_3)]}}\right]. \tag{54}$$

The function $g(t)$ in expression (18) can now be determined, for it equals the perturbed pressure at the edge of the contact region, namely

$$p'_{y=0} = \omega_3 A_3 K\frac{(R/\sqrt{\lambda})+1}{(R/\sqrt{\lambda})-1}\frac{1}{\sqrt{t}}, \tag{55}$$

which is $O(\delta)$. A higher order approximate equation for the temperature in the contact region could now be obtained by substituting $p = P_3+(p')_{y=0}$ in equation (7) with the viscous term neglected.

From equation (53) and the perturbed shock equations we can determine how the shock is affected by the conducting wall. It can be shown that the perturbed shock speed, U'_s, is given by the equation

$$U'_s = \Psi(M_s)\,u'_s, \tag{56}$$

where
$$\Psi(M_s) = \tfrac{1}{2}(\gamma+1)\frac{[(\gamma-1)M_s^2+2]}{[(3\gamma-1)M_s^2+3-\gamma]} \tag{57}$$

and
$$u'_s = -\frac{K}{\sqrt{(1+M_s)}}\frac{R-1}{(R/\sqrt{\lambda})-1}\frac{1}{\sqrt{t}}. \tag{58}$$

The reflected shock is thus attenuated, the perturbed speed varying inversely as the square root of the time.

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